

# On Thermodynamics of AdS Black Holes in Arbitrary Dimensions

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## Abstract

Considering the cosmological constant  $\Lambda$  as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume as proposed in [1], we discuss the critical behavior of charged AdS black hole in arbitrary dimensions  $d$ . In particular, we present a comparative study in terms of the spacetime dimension  $d$  and the displacement of critical points controlling the transition between the small and the large black holes. Such behaviors vary nicely in terms of  $d$ . Among our result in this context consists in showing that the equation of state for a charged RN-AdS black hole predicts an universal number given by  $\frac{2d-5}{4d-8}$ . The three dimensional solution is also discussed.

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Few weeks after the submission of this work to Chinese Physics Letters (CPL) on August 15th, we learned that authors of [arxiv: 1208.6251] found similar results to ours, more notably the derivation of the same universal number given by  $\frac{2d-5}{4d-8}$ .

The research in theories of black holes has scored remarkable progress and has stimulated several interests. The spot is essentially projected on the strength of such black holes in establishing the very deep and fundamental relationship between the (quantum) gravity and the thermodynamics more notably in anti-De Sitter Space. The principal key of this relationship comes from the thermodynamic behavior of black holes, where it appears that some laws of black holes are, in fact, simply the ordinary laws of thermodynamics applied to a system containing a black hole. We refer for instance to [2].

Recently, several efforts have been devoted to study phase transition and critical phenomena for various AdS black hole backgrounds including Reissner-Nordstrom AdS (RN-AdS) black solutions [2, 3, 4, 5, 6, 7, 8]. In particular, the  $P = P(V, T)$  equation of state for a rotating black hole has been dealt with and it has been found that this analysis share similar feature as the Van der Waals  $P - V$  diagram [8, 1].

More recently, these efforts have been pushed further by studying the P-V criticality of RN-AdS black holes with spherical configurations. It has been discussed the behavior of the Gibbs free energy in the fixed charge ensemble. The authors of [1] have also reported on the phase transition in the (P,T)-plane. In particular, it has been given a nice interplay between the behavior of the RN-AdS black hole system and the Van der Waals fluid. More precisely, P-V criticality, Gibbs free energy, first order phase transition and the behavior near the critical points are identified with the liquid-gas system.

The aim of this work is to contribute to these topics by studying such behaviors in arbitrary dimensions. Applying an analogous analysis to [1] for the four dimensional case, we reconsider the critical behavior of charged RN-AdS black holes in arbitrary dimensions of the spacetime. Identifying, the cosmological constant  $\Lambda$  as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume, we discuss such behaviors by giving a comparative study in terms of the dimension and the displacement of critical points. These parameters control the transition between the small and the large black holes. It has been shown also that such behaviors vary nicely in terms of the dimension of the spacetime in which the black hole lives. Our computations predict an universal number given by  $\frac{2d-5}{4d-8}$ . Then, a particular emphasis put on three dimensional case corresponding to BTZ black hole.

To start let us first consider the Einstein-Maxwell-anti-de Sitter action in higher dimensions  $d$ . The latter reads as

$$\mathcal{I} = -\frac{1}{16\pi G} \int_M dx^d \sqrt{-g} [R - F^2 + 2\Lambda] \quad (1)$$

where the field strength  $F$  is a closed 2-form. Indeed, it can always be locally written as  $F = dA$ , where  $A$  is the potential 1-form. In  $d$  dimensions,  $\Lambda$ , identified with  $-\frac{(d-1)(d-2)}{2\ell^2}$ , is the cosmological constant associated with the characteristic length scale  $\ell$ . Varying the above action with respect to the metric tensor leads to the RN-AdS solution given by

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2\Omega_{d-2}^2 \quad (2)$$

where  $d\Omega_{d-2}^2$  indicates the metric on the  $(d-2)$ -dimensional sphere. In this expression, the function  $V(r)$  takes the following general form

$$V(r) = 1 - \frac{2M}{r^{(d-3)}} + \frac{Q^2}{r^{2(d-3)}} + \frac{r^2}{\ell^2}. \quad (3)$$

It is recalled that the parameter  $M$  indicates the ADM mass of the black hole while the parameter  $Q$  represents the total charge. Using the thermodynamic technics, we calculate

the  $d$  dimensional black hole temperature. It is given by

$$T = \frac{1}{\beta} = \frac{1}{4\pi r_+} \left( (d-3) + \frac{(d-1)}{\ell^2} r_+^2 - \frac{(d-3)Q^2}{r_+^{2(d-3)}} \right) \quad (4)$$

where  $r_+$  is the position of the black hole event horizon determined by solving the condition  $V(r_+) = 0$ . In  $d$  dimension, the entropy, as usually, is given by

$$S \sim \frac{A}{4}, \quad (5)$$

where now  $A = \frac{2\pi^{\frac{d-1}{2}} r_+^{d-2}}{\Gamma(\frac{d-1}{2})}$ . The electrical potential  $\Phi$  measured at infinity with respect to horizon takes the following form

$$\Phi = \frac{1}{c} \frac{Q}{r_+^{d-3}} \quad (6)$$

where  $c$  is a constant depending on the dimension  $d$ . It is given by  $c = \sqrt{\frac{2(d-3)}{d-2}}$ .

Using a similar analysis given in [1], we can get the equation of state for a charged AdS black hole  $P = P(V, T)$  in arbitrary dimension  $d$ . For a fixed charge value, the computation leads to

$$P = \frac{(d-2)T}{4r_+} - \frac{(d-3)(d-2)}{16\pi r_+^2} + \frac{(d-3)(d-2)Q^2 r_+^{4-2d}}{16\pi} \quad (7)$$

where  $T$  is the temperature of the black hole. In this expression, the event horizon radius  $r_+$  takes the following general form

$$r_+ = \left( \frac{\Gamma\left(\frac{d+1}{2}\right) V}{\pi^{\frac{d-1}{2}}} \right)^{\frac{1}{d-1}} \quad (8)$$

It is clear that the thermodynamic volume  $V$  is related to the event horizon radius  $r_+$ . It is recalled that  $\Gamma$  is the gamma function. For  $d = 4$ , we recover the value given in [1].

Mimicking the four dimensional analysis given in [1], the physical pressure and temperature become respectively as

$$Press = \frac{\hbar c}{l_P} P, \quad Temp = \frac{\hbar c}{k} T \quad (9)$$

where the Planck length is given by  $l_P^2 = \frac{\hbar G_d}{c^3}$ . The general expression, we are looking for, can be obtained by multiplying (7) by  $\frac{\hbar c}{l_P}$ . Indeed, straightforward calculations lead to the following form

$$Press = \frac{\hbar c}{l_P} \frac{(d-2)T}{4r_+} + \dots = m \frac{k}{2l_P^2 r_+} + \dots \quad (10)$$

A close inspection around the Van der Waals equation given by  $(P + \frac{a}{v^2})(v - b) = kT$  reveals that one can identify the specific volume  $v$  with the following expression

$$v = \frac{4l_P^{2-d}}{(d-2)} r_+. \quad (11)$$

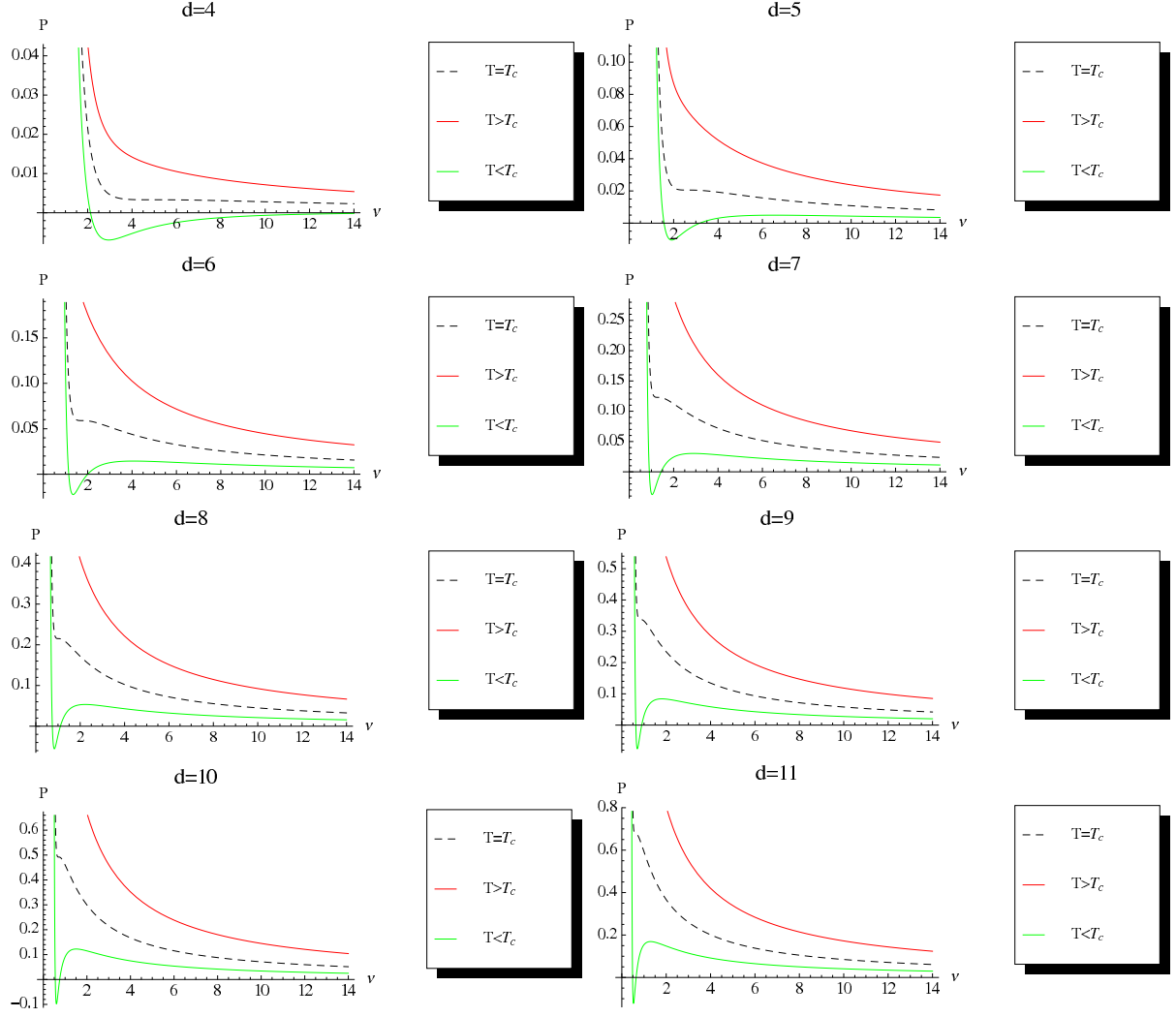


Figure 1: The  $P - V$  diagram of charged AdS black holes in arbitrary dimensions, where  $T_c$  is the critical temperature and the charge is equal to 1

In  $d$  dimensions, the equation of state (7) reads

$$P = \frac{T}{v} - \frac{d-3}{\pi(d-2)v^2} + \frac{(d-3)(d-2)^{(5-2d)}Q^2}{16^{(3-d)}\pi v^{2(d-2)}} \quad (12)$$

Numerical computations drive us to the corresponding "P-V" diagram, plotted in figure1

It follows that for  $Q \neq 0$  and for  $T < T_c$ , the behavior looks like an extended Van der Waals gas and the corresponding system involves inflection points. As in four dimensions, the critical points are solution of the following conditions

$$\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0. \quad (13)$$

At the vicinity of such critical values, we have calculated the following thermodynamic quantities

$d$	$T_c$	$v_c$	$P_c$	$cst$
4	$\frac{1}{3\sqrt{6}\pi Q}$	$2\sqrt{6}Q$	$\frac{1}{96\pi Q^2}$	$\frac{3}{8}$
5	$\frac{4}{5\sqrt[4]{15}\pi\sqrt{Q}}$	$\frac{4\sqrt[4]{5}\sqrt{Q}}{3^{3/4}}$	$\frac{1}{4\sqrt[4]{15}\pi Q}$	$\frac{5}{12}$
6	$\frac{9}{7\sqrt[3]{2}\sqrt[6]{7}\pi\sqrt[3]{Q}}$	$\sqrt[3]{2}\sqrt[6]{7}\sqrt[3]{Q}$	$\frac{9}{162^{2/3}\sqrt[3]{7}\pi Q^{2/3}}$	$\frac{7}{16}$
7	$\frac{16}{9\sqrt[4]{3}\sqrt[8]{5}\pi\sqrt[4]{Q}}$	$\frac{4\sqrt[4]{3}\sqrt[4]{Q}}{5^{7/8}}$	$\frac{1}{\sqrt{3}\sqrt[4]{5}\pi\sqrt{Q}}$	$\frac{9}{20}$
8	$\frac{25}{11^{10}\sqrt[10]{66}\pi\sqrt[5]{Q}}$	$\frac{2^{10}\sqrt[10]{22}\sqrt[5]{Q}}{3^{9/10}}$	$\frac{25}{16\sqrt[5]{66}\pi Q^{2/5}}$	$\frac{11}{24}$
9	$\frac{36}{13^{12}\sqrt[12]{91}\pi\sqrt[6]{Q}}$	$\frac{4^{12}\sqrt[12]{13}\sqrt[6]{Q}}{7^{11/12}}$	$\frac{9}{4\sqrt[6]{91}\pi\sqrt[3]{Q}}$	$\frac{13}{28}$
10	$\frac{49}{152^{3/14}\sqrt[14]{15}\pi\sqrt[7]{Q}}$	$\frac{\sqrt[14]{15}\sqrt[7]{Q}}{2^{11/14}}$	$\frac{49}{162^{3/7}\sqrt[7]{15}\pi Q^{2/7}}$	$\frac{15}{32}$
11	$\frac{64}{17^8\sqrt[8]{3}\sqrt[16]{17}\pi\sqrt[8]{Q}}$	$\frac{4^{16}\sqrt[16]{17}\sqrt[8]{Q}}{33^{7/8}}$	$\frac{4}{\sqrt[4]{3}\sqrt[8]{17}\pi\sqrt[4]{Q}}$	$\frac{17}{36}$

It is observed that the quantity  $\frac{P_c v_c}{T_c}$  varies when the dimension of the black hole is changed. Here, we are very interested in how this change looks like. It is recalled that for  $d = 4$ , this quantity is fixed to  $\frac{3}{8}$  being exactly the same for the Van der Waals fluid and it is considered as an universal number predicted for any charged AdS black hole. We will see that such a number behaves nicely in terms of  $d$ . To see that, we first vary the dimension  $d$ , then we plot the corresponding result in figure 2.

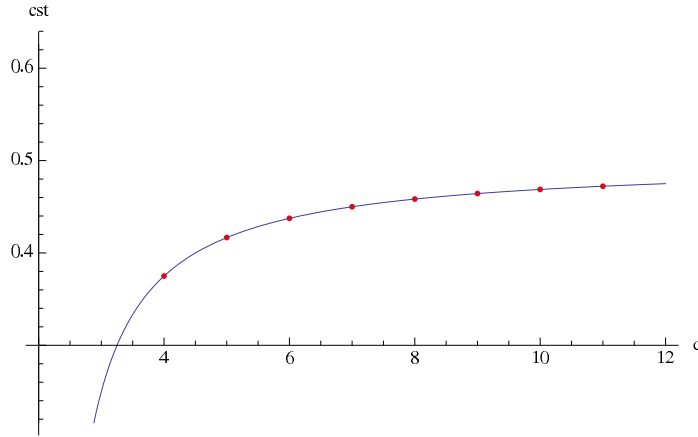


Figure 2: The behavior of the universal constant  $cst$  in terms of  $d$ .

Fitting our result, the behavior of the extended universal quantity takes a very nice form. Indeed, it is given by

$$\frac{P_c v_c}{T_c} = \frac{2d - 5}{4d - 8} \quad (14)$$

This is a nice general expression since it gives, as a particular case for  $d = 4$ , the value  $\frac{3}{8}$  obtained in [1]. This shows that the critical points controlling the transition between the small and the large black hole vary in terms of the dimension of the spacetime. This numerical calculation reveals that the localization of the critical points depends on the dimension  $d$ . Augmenting the dimension of the spacetime such points get dapper. Moreover, it has been observed that small black hole phase becomes relevant for higher dimensional models.

We have to underline that it is remarkably seen that the extended universal constant number  $\frac{2d-5}{4d-8}$  share a striking resemblance with the ideal gas constant  $R$ .

It is worth noting that the critical points obey a nice polynomial shape. The obtained result is presented in figure 3

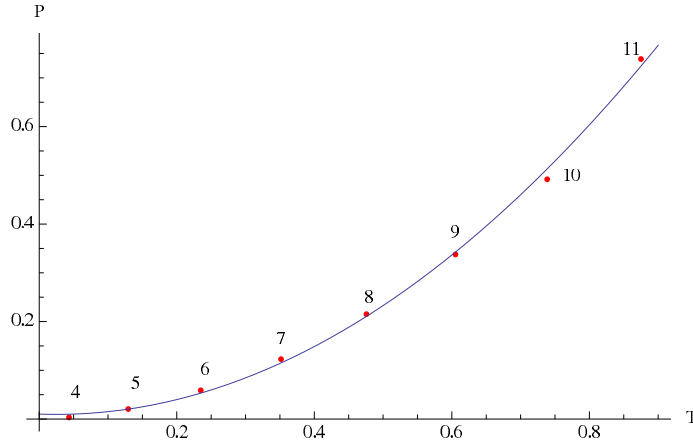


Figure 3: Critical point positions in function of dimensions  $d$

Fitting this numerical calculation, the behavior of such points is controlled by the following polynomial function

$$P = \alpha + \beta T + \gamma T^2 \quad (15)$$

For a particular choice in the parameter space where the charge  $Q$  is fixed to 1, the above coefficients are given by

$$\alpha = 0.0102026, \quad \beta = 0.0492466 \quad \gamma = 0.988634. \quad (16)$$

It will be interesting to understand the polynomial behavior of the critical points in terms of the dimension.

In the end of this work, we would like to make comment on a particular case corresponding to  $d = 3$ . In this case, the equation (7) reduces to

$$Pv = T \quad (17)$$

describing an ideal gas equation. Treating the dimension as a size parameter, it seems that the Van der Waals analysis can be converted to an ideal gas model. On the other hand, it

is recalled that in  $d = 3$  lives a BTZ black hole solution[9]. Following [10, 11], the BTZ temperature is given by

$$T = \frac{1}{4\pi} \left( \frac{2r_+}{\ell^2} - \frac{\pi Q^2}{r_+} \right). \quad (18)$$

The corresponding state equation can be written as

$$P = \frac{Q^2}{8\pi r_+^2} + \frac{T}{4r_+}. \quad (19)$$

This looks like (7) without critical points. It follows that BTZ black hole satisfies an ideal gas behavior. This is can be obtained by turning off the coupling Maxwell gauge fields.

Note by the way, that the state equation corresponding to a  $d$  dimensional BTZ black hole reads as

$$P = \frac{(d-2)T}{4r_+} + \frac{2^{\frac{d-9}{2}}(d-2)Q^{d-1}r_+^{1-d}}{\pi}. \quad (20)$$

Finally, let us mention that the above behavior present only  $d > 2$ . For  $d = 2$ , the equation of state becomes  $P = 0$ . It should be interesting to reconsider the discussion of such a case in connection with  $2d$  black holes and Liouville theory. This will be addressed in future works.

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